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ON REALIZING NONHOLONOMIC CONSTRAINTS BY VISCOUS FRICTION FORCES AND CELTIC STONES STABILITY*

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It is shown that equations of motion of nonholonomic systems can be obtained from the equations of motion of systems freed of nonholonomic constraints subjected to suitably chosen dissipative forces, if in the latter the dissipation coefficient is assumed infinite. It is shown that on fairly general assumptions motions of nonholonomic systems represent limits of motions of corresponding holonomic systems, as the dissipation coefficient approaches infinity. The stability of rotation on a horizontal plane of a heavy asymmetric rigid body (Celtic stone) about the vertical, with allowance for friction, is investigated. The obtained stability conditions are compared with previously published papers about rotation of a body on an absolutely rough horizontal plane around the vertical.

1. Consider a mechanical system whose position is determined by n generalized coordinates $q_1, \ldots, q_n$ and its dynamic properties are defined by the Lagrange functions $L(q, q^{\cdot}) = T(q, q^{\cdot}) + U(q)$ and generalized forces $Q_s(q, q^{\cdot})$ $(s = 1, \ldots, n)$. We assume the kinetic energy T to be a positve definite quadratic form of velocities $q_1^{\cdot}, \ldots, q_n^{\cdot}$ of generalized coordinates, whose coefficients and the force function U are twice continuously differentiable with respect to q, and the generalized forces Q_s are continuously differentiable with respect to q and $q^{\cdot}$.

Let us assume that the system is subjected to nonintegrable constraints of the form

$$q_\alpha^{\cdot} = \sum_{i=1}^{m} b_{\alpha i}(q) q_i^{\cdot} \quad (\alpha = m+1, \ldots, n) \tag{1.1}$$

whose coefficients $b_{\alpha i}$ are also twice continuously differentiable with respect to q. The system is then nonholonomic and its motion can be defined, for instance, by Voronets' equations

$$\frac{d}{dt}\frac{\partial \Theta}{\partial q_i^{\cdot}} = \frac{\partial(\Theta + U)}{\partial q_i} + \Pi_i + \sum_{\alpha=m+1}^{n}\left[\frac{\partial(\Theta+U)}{\partial q_\alpha} + \Pi_\alpha\right] b_{\alpha i} + \sum_{\alpha=m+1}^{n} \Theta_\alpha \sum_{j=1}^{m} \nu_{\alpha ij} q_j^{\cdot} \quad (i = 1, \ldots, m) \tag{1.2}$$

$$\nu_{\alpha ij} = \frac{\partial b_{\alpha i}}{\partial q_j} - \frac{\partial b_{\alpha j}}{\partial q_i} + \sum_{\beta=m+1}^{n}\left(\frac{\partial b_{\alpha i}}{\partial q_\beta} b_{\beta j} - \frac{\partial b_{\alpha j}}{\partial q_\beta} b_{\beta i}\right)$$

which with the addition of Eq.(1.1) of constraints form a closed system. In these equations Θ, Θ_α, and Π_s are obtained, respectively, from T, $\partial T/\partial q_\alpha^{\cdot}$, and Q_s by eliminating $q_\alpha^{\cdot}$ using relations (1.1).

Under these conditions Eqs.(1.1) and (1.2) constitute a closed system of the $(n+m)$-th order in $(n+m)$ unknowns $q_1, \ldots, q_n, q_1^{\cdot}, \ldots, q_m^{\cdot}$, which satisfies the theorems of existence and uniqueness of solution.

Let now all coordinates q_s and velocities $q_s^{\cdot}$ of the system be independent (free of constraints (1.1)), i.e. the system is holonomic, and let it be subjected, besides the input forces, to dissipative forces $F_s = -\partial F/\partial q_s^{\cdot}$ $(s = 1, \ldots, n)$ which are derivatives of a function of the form

$$F = \frac{1}{2} k \sum_{\alpha=m+1}^{n}\left(q_\alpha^{\cdot} - \sum_{i=1}^{m} b_{\alpha i} q_i^{\cdot}\right)^2, \quad k > 0 \tag{1.3}$$

We take the equations of motion in the Lagrange form

$$\frac{d}{dt}\frac{\partial L}{\partial q_s^{\cdot}} = \frac{\partial L}{\partial q_s} + Q_s - \frac{\partial F}{\partial q_s^{\cdot}} \quad (s = 1, \ldots, n) \tag{1.4}$$

Under the indicated conditions (1.4) represents a closed system of the $2n$-th order in $2n$ unknowns $q_1, \ldots, q_n, q_1^{\cdot}, \ldots, q_n^{\cdot}$, which satisfies the theorem of existence and uniqueness of solution.

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Eliminating k from the first m equations of system (1.4) using the last $(n-m)$ equations and dividing the last $(n-m)$ equations of that system by k, we obtain

$$\frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}} - \frac{\partial L}{\partial q_i} - Q_i + \sum_{\alpha=m+1}^{n}\left(\frac{d}{dt}\frac{\partial L}{\partial q_\alpha^{\cdot}} - \frac{\partial L}{\partial q_\alpha} - Q_\alpha\right) b_{\alpha i} = 0 \qquad (1.5)$$

$$(i = 1, \ldots, m)$$

$$\frac{1}{k}\left(-\frac{d}{dt}\frac{\partial L}{\partial q_\alpha^{\cdot}} + \frac{\partial L}{\partial q_\alpha} + Q_\alpha\right) = q_\alpha^{\cdot} - \sum_{j=1}^{m} b_{\alpha j} q_j^{\cdot} \quad (\alpha = m+1, \ldots, n) \qquad (1.6)$$

If we set now $k=\infty$, system (1.6) assumes the form of Eq. (1.1) of constraints and system (1.5), after the elimination from it $q_\alpha^{\cdot}$ using formulas (1.6) with $k=\infty$ (i.e. (1.1)), assumes the form of Voronets' equations (1.2).

The equations of motion of a nonholonomic system with constraints of form (1.1) are, thus, obtained from the equations of motion of a system freed of nonholonomic constraints subjected in addition to the input forces to dissipative ones that are derivatives of a function of form (1.3) by setting in the latter k equal infinity.

Remark. Roling of a rigid body over an absolutely rough surface (without slippage) generates nonholonomic constraints of form (1.1) which express the condition of zero velocity of the contact point between body and supporting surface. If the system is freed from such nonholonomic constraints (admission of the possibility of slippage) and dissipation of form (1.3) is introduced, then to the latter correspond forces acting on the body at its contact point with the supporting surface, which are proportional to the velocity of that point of the body, in the opposite direction, i.e. viscous friction forces.

2. We shall now prove that Tikhonov's theorem /1,2/ provides a positive answer to the question of closeness of the nonholonomic system motion to satisfying the initial conditions $q_{s0}\ (s=1,\ldots,n)$, $q_{i0}^{\cdot}\ (i=1,\ldots,m)$, and to the holonomic system with initial conditions $q_{s0}\ (s=1,\ldots,n)$, $q_{s0}^{\cdot}\ (s=1,\ldots,n)$, as $k\to\infty$.

Note that the last $n-m$ of initial velocities of the holonomic system $q_{\alpha 0}^{\cdot}\ (\alpha=m+1,\ldots,n)$ are arbitrary and may not satisfy formulas (1.1) at the initial instant.

We introduce the quasi-momenta

$$\pi_i = p_i + \sum_{\alpha=m+1}^{n} p_\alpha b_{\alpha i} \quad (i=1,\ldots,m), \quad \pi_\alpha = p_\alpha \quad (\alpha=m+1,\ldots,n) \qquad (2.1)$$

where $p_s = \partial L/\partial q_s^{\cdot}\ (s=1,\ldots,n)$ are momenta that correspond to velocities $q_s^{\cdot}$ of the system freed of nonholonomic constraints, and rewrite system (1.5), (1.6) in the canonical form

$$q_i^{\cdot} = \frac{\partial H^*}{\partial \pi_i} \quad (i=1,\ldots,m) \qquad (2.2)$$

$$q_\alpha^{\cdot} = \frac{\partial H^*}{\partial \pi_\alpha} + \sum_{i=1}^{m}\frac{\partial H^*}{\partial \pi_i} b_{\alpha i} \quad (\alpha = m+1,\ldots,n) \qquad (2.3)$$

$$\pi_i^{\cdot} = -\frac{\partial H^*}{\partial q_i} + P_i^* - \sum_{\alpha=m+1}^{n}\left(\frac{\partial H^*}{\partial q_\alpha} - P_\alpha^*\right) b_{\alpha i} + \sum_{\alpha,\beta=m+1}^{n} \pi_\alpha \frac{\partial b_{\alpha i}}{\partial q_\beta}\frac{\partial H^*}{\partial \pi_\beta} + \sum_{\alpha=m+1}^{n} \pi_\alpha \sum_{j=1}^{m} \nu_{\alpha i j}\frac{\partial H^*}{\partial \pi_j} \quad (i=1,\ldots,m) \qquad (2.4)$$

$$\frac{1}{k}\pi_\alpha^{\cdot} = -\frac{\partial H^*}{\partial \pi_\alpha} - \frac{1}{k}\left(\frac{\partial H^*}{\partial q_\alpha} - P_\alpha^* - \sum_{\beta=m+1}^{n}\pi_\beta \sum_{j=1}^{m}\frac{\partial b_{\beta j}}{\partial q_\alpha}\frac{\partial H^*}{\partial \pi_j}\right), \quad (\alpha=m+1,\ldots,n) \qquad (2.5)$$

where $H^*(q,\pi)$ is obtained from the Hamiltonian $H(q,p)$ that corresponds to the Lagrangian $L(q,q^{\cdot})$ by eliminating $p_s\ (s=1,\ldots,n)$ using formulas (2.1); similarly $P_s^*\ (s=1,\ldots,n)$ are obtained from P_s of generalized forces Q_s, expressed in terms of coordinates and momenta. The case of $k=\infty$ in which system (2.5) assumes the form

$$-\partial H^*/\partial \pi_\alpha = 0 \quad (\alpha = m+1,\ldots,n) \qquad (2.6)$$

corresponds to a nonholonomic system.

Systems (2.2)—(2.5) and (2.2)—(2.4), (2.6) evidently satisfy on the above assumptions, the theorems of existence and uniqueness of solution, i.e. to the first and third conditions of Tikhonov's theorem /1,2/. Moreover, system (2.6) has the unique solution

$$\pi_\alpha = \varphi_\alpha(q_1,\ldots,q_n,\ \pi_1,\ldots,\pi_m) \quad (\alpha=m+1,\ldots,n) \qquad (2.7)$$

that corresponds to the equations of nonholonomic constraints. Since

$$H^*(q_s, \pi_i, \pi_\alpha) = H\left(q_s, p_i - \sum_\beta p_\beta b_{\beta i}, p_\alpha\right)$$

system (2.6) is of the form

$$\sum_{i=1}^{m} \frac{\partial H}{\partial p_i} b_{\alpha i} - \frac{\partial H}{\partial p_\alpha} = 0 \quad (\alpha = m+1, \ldots, n)$$

i.e. ($q_s^{\cdot} = \partial H / \partial p_s$) is of the form (1.1), By virtue of the uniqueness and differentiability under the above assumptions, solution (2.7) satisfies the second of Tikhonov's theorems /1,2/.

Let us consider the system

$$\pi_\alpha' = -\partial H^* / \partial \pi_\alpha \quad (\alpha = m+1, \ldots, n) \tag{2.8}$$

where the prime denotes differentiation with respect to some independent variable τ, and $q_s (s = 1, \ldots, n)$ and $\pi_i (i = 1, \ldots, m)$ are considered as parameters. The solution (*) (2.7) of system (2.8) is asymptotically stable uniformly with respect to q_s and π_i, since the equations of perturbed motion are of the form

$$z_\alpha' = -\sum_{\beta=m+1}^{n} a_{\alpha\beta}(q) z_\beta \quad (z_\alpha = \pi_\alpha - \varphi_\alpha(q_s, \pi_i), \alpha = m+1, \ldots, n) \tag{2.9}$$

where $\| a_{\alpha\beta}(q) \|$ is a matrix of the coefficients of the quasi-Hamiltonian H^* for products of the last $n-m$ quasi-momenta, whose all eigenvalues are positive (the kinetic energy is by assumption positive definite relative to velocities, and this implies the positive definiteness of the Hamiltonian relative to momenta, as well as that of the quasi-Hamiltonian relative to quasi-momenta). It is also obvious that the solution of system (2.5) with any initial $z_{\alpha 0}$ approaches asymptotically zero, i.e. the whole domain of variation of π_α is the region of attraction of point (2.7) of system (2.8).

Consequently the last conditions of Tikhonov's theorem /1,2/ are satisfied, i.e. there exists a $0 < k_0 < \infty$ such that for $k > k_0$ the solutions of system (2.2)—(2.5) with initial conditions q_{s0}, π_{s0} approach in any finite time interval the solution of system (2.2)—(2.4), (2.6) with initial conditions q_{s0}, π_{i0}; as $k \to \infty$. In that case $\pi_{\alpha 0}$ may not satisfy system (2.6).

3. The following theorem is thus proved.

Theorem. Let the force function and kinetic energy coefficients of the quadratic form of velocities, assumed positive definite, be continuously twice differentiable with respect to coordinates, and the forces acting on the system be continuously differentiable with respect to coordinates and velocities. Then the motions of a system subjected to nonholonomic constraints of form (1.1) with coefficients twice continuously differentiable with respect to coordinates are limit motions in any finite time interval for the corresponding motions of a system freed of nonholonomic constraints and subjected (in addition to input) to dissipative forces that are derivatives of functions of form (1.3), as the dissipation coefficient approaches infinity.

Remarks. 1°. The initial conditions of a free system may not satisfy the equations of constraints. Consequently, the state of a system free of nonholonomic constraints at the initial instant can substantially differ from that of a system with constraints. However the difference is small for times of order $\ln k/k$ and approaches zero as $k \to \infty$ /2/.

2°. With appropriate strengthening of requirements as regards the differentiability of Lagrangian functions, generalized forces, and coefficients of constraints the asymptotic expansions in the small parameter $1/k$ /2/ are valid for solving systems with dissipations freed of constraints.

The proved theorem means in particular that the nonholonomic constraints that arise when bodies move over absolutely rough surfaces (without slippage) can be due to viscous friction forces when the friction coefficient is infinite.

A similar statement about a specific system, that of Chaplygin's sledge on a horizontal plane in the presence of nonholonomic constraint or of viscous friction force, was earlier obtained by Fufaev /3/ in the course of investigation of integral curves.

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*) Editor's note: In the Russian original the word "equilibrium (2.7)" is used.

Note that viscous friction occurs when a body moves on a surface in a sliding or rotation mode /4,5/ (in pure sliding mode, dry friction must be considered).

4. Necessary and sufficient conditions of stability of rotation of a heavy solid body about the vertical on an absolutely rough horizontal plane were established in /6,7/. We shall now investigate the case of a smooth plane with viscous friction, previously investigated only numerically in /8/.

Consider a heavy solid body bounded by a convex surface on a horizontal plane, assuming that at the point of contact of the body and plane a force proportional to the velocity of the point of contact of the body with the plane acts in the opposite direction (the viscous friction force) on the body. We specify the position of the body by the coordinates x, y of the body center of mass in the fixed system of coordinates $Oxyz$ (with the Oxy-plane coinciding with the horizontal plane and the Oz-axis directed vertically upward) and by Euler's angles θ, φ, ψ of the principal central axes $G\xi$, $G\eta$, $G\zeta$ of the ellipsoid of inertia of the body relative to the fixed coordinate system.

The Lagrangian functions of the system and the Rayleigh dissipative function assumes the form

$$L = {}^1/_2\,[A\cos^2\varphi + B\sin^2\varphi + m(\gamma_1\cos\theta - \zeta\sin\theta)^2]\,\theta^{\cdot 2} + {}^1/_2\,(C + m\gamma_2^2\sin^2\theta)\,\varphi^{\cdot 2} + {}^1/_2\,[(A\sin^2\varphi + B\cos^2\varphi)\sin^2\theta + C\cos^2\theta]\psi^{\cdot 2} + m(\gamma_1\cos\theta - \zeta\sin\theta)\,\gamma_2\sin\theta\theta^{\cdot}\varphi^{\cdot} + (A-B)\sin\theta\sin\varphi\cos\varphi\theta^{\cdot}\psi^{\cdot} + C\cos\theta\varphi^{\cdot}\psi^{\cdot} + {}^1/_2\,m(x^{\cdot 2} + y^{\cdot 2}) + mg(\gamma_1\sin\theta + \zeta\cos\theta)$$

$$F = {}^1/_2\,mk\,\{[x^{\cdot} - (\alpha_1\theta^{\cdot} + \alpha_2\varphi^{\cdot} + \alpha_3\psi^{\cdot})]^2 + [y^{\cdot} - (\beta_1\theta^{\cdot} + \beta_2\varphi^{\cdot} + \beta_3\psi^{\cdot})]^2\}$$

$$\alpha_1 = -\sin\psi\,(\gamma_1\sin\theta + \zeta\cos\theta),\quad \alpha_2 = \gamma_1\cos\psi + \gamma_2\cos\theta\sin\psi$$

$$\alpha_3 = \gamma_2\sin\psi + (\gamma_1\cos\theta - \zeta\sin\theta)\cos\psi,\quad \beta_i = -\partial\alpha_i/\partial\psi$$

$$(i = 1, 2, 3)$$

$$\gamma_1 = \xi\sin\varphi + \eta\cos\varphi,\quad \gamma_2 = \xi\cos\varphi - \eta\sin\varphi$$

where m is the mass of the body, A, B, C are its principal central moment of inertia, $k > 0$ is the friction coefficient, and ξ, η, ζ are coordinates of the contact point of the body with the plane in the system $G\xi\eta\zeta$. It can be shown that ξ, η, ζ are functions of variables θ and φ that are determined by the form of the equation defining the body surface, and satisfy relations of the form

$$(\xi'\sin\varphi + \eta'\cos\varphi)\sin\theta + \zeta'\cos\theta \equiv 0 \tag{4.1}$$

where the prime denotes differentiation with respect to θ or φ.

Since in input coordinates F is explicitly dependent on ψ, we substitute for the $x\,y$ coordinates the quasicoordinates ρ and σ defined by formulas

$$\rho^{\cdot} = x^{\cdot}\sin\psi - y^{\cdot}\cos\psi,\quad \sigma^{\cdot} = x^{\cdot}\cos\psi + y^{\cdot}\sin\psi \tag{4.2}$$

We denote functions L and F, after elimination from them of $x^{\cdot}$ and $y^{\cdot}$ using (4.2), by L^* and F^* which depend only on $\theta, \varphi, \theta^{\cdot}, \varphi^{\cdot}, \psi^{\cdot}, \rho^{\cdot}, \sigma^{\cdot}$.

In the new variables the equations of motion

$$\frac{d}{dt}\frac{\partial L^*}{\partial \chi_i^{\cdot}} = \frac{\partial L^*}{\partial \chi_i} - \frac{\partial F^*}{\partial \chi_i^{\cdot}}\quad (i = 1, 2, 3;\ \ \chi_1 = \theta,\ \ \chi_2 = \varphi,\ \ \chi_3 = \psi) \tag{4.3}$$

$$\frac{d}{dt}\frac{\partial L^*}{\partial \rho^{\cdot}} = -\frac{\partial F^*}{\partial \rho^{\cdot}} + \frac{\partial L^*}{\partial \sigma^{\cdot}}\psi^{\cdot},\quad \frac{d}{dt}\frac{\partial L^*}{\partial \sigma^{\cdot}} = -\frac{\partial F^*}{\partial \sigma^{\cdot}} - \frac{\partial L^*}{\partial \rho^{\cdot}}\psi^{\cdot}$$

do not explicitly contain ψ, ρ, σ, only their velocities and accelerations appear in them. It is reasonable to call the variables ignorable /9/, and refer to θ and φ as positional, and formulate the problem of determining steady motions of the form

$$\theta = \theta_0,\ \theta^{\cdot} = 0,\ \varphi = \varphi_0,\ \varphi^{\cdot} = 0,\ \psi^{\cdot} = \psi_0^{\cdot},\ \rho^{\cdot} = \rho_0^{\cdot},\ \sigma^{\cdot} = \sigma_0^{\cdot} \tag{4.4}$$

and of their stability.

Substituting (4.4) into the equations of motion (4.3) with allowance for formulas (4.1), we obtain that $\rho_0^{\cdot} = \sigma_0^{\cdot} = 0$, $\psi_0^{\cdot}$ is arbitrary, and θ_0 and φ_0 are such that one of the principal axes of the body ellipsoid of inertia lies on the veritical passing through the point of contact between body and the horizontal plane. System (4.3) admits the solution

$$\theta = \pi/2,\ \theta^{\cdot} = 0,\ \varphi = \varphi^{\cdot} = 0,\ \psi^{\cdot} = \omega = \text{const},\ \rho^{\cdot} = \sigma^{\cdot} = 0 \tag{4.5}$$

which corresponds to rotation of a body about the vertically positioned principal axis $G\eta$ of the body ellipsoid of inertia at constant angular velocity.

5. Let us investigate the stability of solution (4.5) with respect to perturbations of variables $\theta, \theta^{\cdot}, \varphi, \varphi^{\cdot}, \psi^{\cdot}, \rho^{\cdot}, \sigma^{\cdot}$.

After transformations, the equations of perturbed motion are reduced to the form

$$\begin{aligned}
&Au^{\cdot\cdot} + (A + C - B)\,\omega v^{\cdot} + [(B - C)\,\omega^2 + mg\,(r_1\cos^2\alpha + \\
&\quad r_2\sin^2\alpha - a)]\,u + mg\,(r_2 - r_1)\sin\alpha\cos\alpha v + mar^{\cdot} - ma\omega s = U \\
&Cv^{\cdot\cdot} - (A + C - B)\,\omega u^{\cdot} + [(B - A)\,\omega^2 + mg\,(r_1\sin^2\alpha + \\
&\quad r_2\cos^2\alpha - a)]\,v + mg\,(r_2 - r_1)\sin\alpha\cos\alpha u - mas^{\cdot} - ma\omega r = V,\ w^{\cdot} = W \\
&r^{\cdot} + kr - kau^{\cdot} - k\omega\,[-(r_2 - r_1)\sin\alpha\cos\alpha\, u + (a - r_1\sin^2\alpha - r_2\cos^2\alpha)\,v] - \omega s = R \\
&s^{\cdot} + ks + kav^{\cdot} - k\omega\,[(a - r_1\cos^2\alpha - r_2\sin^2\alpha)\,u - (r_2 - r_1)\sin\alpha\cos\alpha v] + \omega r = S
\end{aligned} \tag{5.1}$$

where u, v, w, r, s are perturbations of variables $\theta, \varphi, \psi^{\cdot}, \rho^{\cdot}, \sigma^{\cdot}$, U, V, W, R, S are functions of variables $u, u^{\cdot}, v, v^{\cdot}, w, r, s$, whose expansion begins with terms of order not lower than the second, with U_0, V_0, W_0, R_0, S_0 identically zero (the zero subscript indicates that in respective functions all variables, except w, are assumed equal zero); a is the distance of the point of contact between body and plane to the body center of mass; r_1 and r_2 are the principal radii of the body surface curvature at the point of its contact with the plane, and α is the angle between the principal central axis of the ellipsoid of inertia which corresponds to moment $C\,(G\zeta)$ and the direction of the principal radius of curvature r_1, measured from the $G\zeta$-axis to the $G\xi$-axis.

The characteristic equations for the linearized system is of the form

$$\begin{aligned}
&\lambda f(\lambda) = 0,\ f(\lambda) = J\lambda^6 + K\lambda^5 + L\lambda^4 + M\lambda^3 + N\lambda^2 + P\lambda + Q \\
&J = AC \equiv J_{0,0},\quad K = [2AC + (A + C)\,ma^2]\,k \equiv K_{0,1}k \\
&L = [2AC + (B - A)(B - C)]\,\omega^2 + (A - C)\,ma\,(r_2 - r_1)\times \\
&\quad \sin\alpha\cos\alpha\, k\omega + (A + ma^2)(C + ma^2)\,k^2 + mg\times \\
&\quad [A\,(r_1\sin^2\alpha + r_2\cos^2\alpha - a) + C\,(r_1\cos^2\alpha + r_2\sin^2\alpha - a)] \equiv L_{2,0}\omega^2 + L_{1,1}\omega k + L_{0,2}k^2 + L_{0,0} \\
&M = [2AC + 2(B - A)(B - C) + 2(A + C)\,ma^2 + \\
&\quad (B - A - C)\,ma\,(r_1 + r_2)]\,\omega^2 k + (A - C)\,ma\,(r_2 - r_1)\times \\
&\quad \sin\alpha\cos\alpha\,\omega k^2 + mg\,[2A\,(r_1\sin^2\alpha + r_2\cos^2\alpha - a) + \\
&\quad 2C\,(r_1\cos^2\alpha + r_2\sin^2\alpha - a) + ma^2\,(r_1 + r_2 - 2a)]k \equiv (M_{2,1}\omega^2 + M_{1,2}\omega k + M_{0,1})\,k \\
&N = [AC + 2(B - A)(B - C)]\omega^4 + 2(A - C)\,ma\,(r_2 - \\
&\quad r_1)\sin\alpha\cos\alpha\,\omega^3 k + \{AC + (B - A)(B - C) + \\
&\quad ma\,[B\,(r_1 + r_2 - 2a) - A\,(r_1\cos^2\alpha + r_2\sin^2\alpha - 2a) - \\
&\quad C\,(r_1\sin^2\alpha + r_2\cos^2\alpha - 2a)] + m^2a^2\,[a^2 + (a - r_1)\times \\
&\quad (a - r_2)]\}\,\omega^2k^2 + mg\,[B\,(r_1 + r_2 - 2a) + (A - C)(r_2 - r_1)\times \\
&\quad (\cos^2\alpha - \sin^2\alpha)]\,\omega^2 + mg\,[A\,(r_1\sin^2\alpha + r_2\cos^2\alpha - a) + \\
&\quad C\,(r_1\cos^2\alpha + r_2\sin^2\alpha - a) + ma^2\,(r_1 + r_2 - 2a)]k^2 + \\
&\quad m^2g^2\,(a - r_1)(a - r_2) \equiv N_{4,0}\omega^4 + N_{3,1}\omega^3k + N_{2,2}\omega^2k^2 + N_{2,0}\omega^2 + N_{0,2}k^2 + N_{0,0} \\
&P = [2(B - A)(B - C) + (B - A - C)\,ma\,(r_1 + r_2) + \\
&\quad (A + C)\,ma^2]\,\omega^4k + (A - C)\,ma\,(r_2 - r_1)\sin\alpha\cos\alpha\,\omega^3k^2 + \\
&\quad mg\,[2B\,(r_1 + r_2 - 2a) - 2A\,(r_1\cos^2\alpha + r_2\sin^2\alpha - a) - \\
&\quad 2C\,(r_1\sin^2\alpha + r_2\cos^2\alpha - a) + ma^2\,(r_1 + r_2 - 2a)]\,\omega^2k + \\
&\quad 2m^2g^2\,(a - r_1)(a - r_2)\,k \equiv (P_{4,1}\omega^4 + P_{3,2}\omega^3k + P_{2,1}\omega^2 + P_{0,1})\,k \\
&Q = (B - A)(B - C)\,\omega^6 + (A - C)\,ma\,(r_2 - r_1)\times \\
&\quad \sin\alpha\cos\alpha\,\omega^5k + \{(B - A)(B - C) + ma\,[B\,(r_1 + r_2 - 2a) - A\,(r_1\cos^2\alpha + r_2\sin^2\alpha - a) - \\
&\quad C\,(r_1\sin^2\alpha + r_2\cos^2\alpha - a) + ma\,(a - r_1)\,(a - r_2)]\}\,\omega^4k^2 + \\
&\quad [B\,(r_1 + r_2 - 2a) - A\,(r_1\cos^2\alpha + r_2\sin^2\alpha - a) - \\
&\quad C\,(r_1\sin^2\alpha + r_2\cos^2\alpha - a)]\,mg\omega^4 + mg\,[B\,(r_1 + r_2 - 2a) - \\
&\quad A\,(r_1\cos^2\alpha + r_2\sin^2\alpha - a) - C\,(r_1\sin^2\alpha + r_2\cos^2\alpha - a) + \\
&\quad 2ma\,(a - r_1)\,(a - r_2)]\,\omega^2k^2 + m^2g^2\,(a - r_1)\,(a - r_2)\,\omega^2 + \\
&\quad m^2g^2\,(a - r_1)\,(a - r_2)\,k^2 \equiv Q_{6,0}\omega^6 + Q_{5,1}\omega^5k + Q_{4,2}\omega^4k^2 + Q_{4,0}\omega^4 + Q_{2,2}\omega^2k^2 + Q_{2,0}\omega^2 + Q_{0,2}k^2
\end{aligned} \tag{5.2}$$

The characteristic equation (5.2) has obviously one zero root. If in addition at least one root of the equation

$$f(\lambda) = 0 \tag{5.3}$$

lies in the right-hand half-plane, solution (4.5) is unstable, while when all roots of Eq. (5.3) lie in the left-hand half-plane, the Liapunov theorem holds, since system (5.1) is of the form that corresponds to the critical case of a single zero root, and solution (4.5) is stable asymptotically with respect to $\theta, \theta^\cdot, \varphi, \varphi^\cdot, \rho^\cdot, \sigma^\cdot$ and nonasymptotically with respect to $\psi^\cdot$.

6. All roots of Eq. (5.3) lie in the left half-plane then and only then when the following conditions are satisfied /10/:

$$\begin{gathered}L_{2,0}\omega^2 + L_{1,1}\omega k + L_{0,2}k^2 + L_{0,0} > 0 \\ N_{4,0}\omega^4 + N_{3,1}\omega^3 k + N_{2,2}\omega^2 k^2 + N_{2,0}\omega^2 + N_{0,2}k^2 + N_{0,0} > 0 \\ Q_{6,0}\omega^6 + Q_{5,1}\omega^5 k + Q_{4,2}\omega^4 k^2 + Q_{4,0}\omega^4 + Q_{2,2}\omega^2 k^2 + Q_{2,0}\omega^2 + Q_{0,2}k^2 > 0 \\ \delta_{4,2}\omega^4 + \delta_{3,3}\omega^3 k + \delta_{2,4}\omega^2 k^2 + \delta_{1,5}\omega k^3 + \delta_{2,2}\omega^2 + \delta_{1,3}\omega k + \delta_{0,4}k^2 + \delta_{0,2} > 0 \\ \Delta_{12,3}\omega^{12} + \ldots + (\Delta_{6,9}\omega^6 + \Delta_{4,9}\omega^4 + \Delta_{2,9}\omega^2)k^6 + \ldots + \Delta_{0,3} > 0\end{gathered} \tag{6.1}$$

where $\delta_{\mu,\nu}$ and $\Delta_{\mu,\nu}$ are coefficients of the expansion of determinants δ and Δ in powers of ω and k

$$\Delta = \begin{vmatrix} J & L & N & Q & 0 \\ 0 & J & L & N & Q \\ 0 & 0 & K & M & P \\ 0 & K & M & P & 0 \\ K & M & P & 0 & 0 \end{vmatrix}$$

and determinant δ is obtained from determinant Δ by eliminating the first and last rows and columns.

The analysis of conditions (6.1) is considerably simplified when $k \gg |\omega|$ or $|\omega| \gg k$. If $k \gg |\omega|$, conditions (6.1) are equivalent to conditions

$$\begin{gathered}Q_{4,2}\omega^4 + Q_{2,2}\omega^2 + Q_{0,2} > 0 \\ (N_{2,2}\omega^2 + N_{0,2} - L_{0,2}\omega^2)\,\omega^2 - (Q_{4,2}\omega^4 + Q_{2,2}\,\omega^2 + Q_{0,2}) > 0 \\ (A - C)(r_2 - r_1)\sin\alpha\cos\alpha\,\omega > 0\end{gathered} \tag{6.2}$$

which exactly coincide with the conditions of rotation stability of a heavy solid body on an absolutely rough horizontal plane. The first two of conditions (6.2) impose constraints only on the distribution of masses and the body surface geometry, and on the magnitude of, angular velocity, while the third represent constraints on the body direction of rotation.

Thus, if the friction coefficient is fairly large, the conditions of rotation stability of a body on a plane with friction are virtually equivalent to those of rotation stability of a body on an absolutely rough plane.

Note that by dividing Eq. (5.3) by k^2 and then setting $k = \infty$ we reduce it to the form

$$L_{0,2}\lambda^4 + M_{1,2}\omega\lambda^3 + (N_{2,2}\omega^2 + N_{0,2})\lambda^2 + P_{3,2}\omega^3\lambda + (Q_{4,2}\omega^4 + Q_{2,2}\omega^2 + Q_{0,2}) = 0$$

which is exactly the form of the equation for nonzero roots /7/ that corresponds to the case of an absolutely rough plane.

Let now $|\omega| \gg k$, then conditions (6.1) are virtually the same as conditions

$$L_{2,0} > 0,\ N_{4,0} > 0,\ Q_{6,0} > 0,\ \delta_{4,2} > 0,\ \Delta_{12,3} > 0 \tag{6.3}$$

which impose constraints only on the distribution of masses and the body surface geometry. Note that region (6.3) is nonempty in the space of parameters $(A, B, C, m, a, r_1, r_2, \alpha)$ in the neighborhood of the manifold $A = C$, $r_1 = r_2$ which corresponds to the case of rotation of a dynamically symmetirc body about its axis of symmetry, supported by a spherical bearing. This means that when the rotation of the body is fairly fast, its stability in certain particular cases of mass distribution and body surface geometry is independent of the direction of rotation.

However, even in the case of fast rotation, there exists in the space of parameters of the system a region in which the stability of the body rotation substantially depends on the rotation direction. Let, for instance, $B = A \neq C$ or $B = C \neq A$, then $Q_{6,0} = 0$ and when $|\omega| \gg k$ conditions (6.1) are equivalent to the conditions

$$L_{2,0} > 0,\quad N_{4,0} > 0,\quad \delta_{4,2} > 0,\quad \Delta_{12,3} > 0,\quad Q_{5,1}\omega \equiv (A - C)(r_2 - r_1)\sin\alpha\cos\alpha\omega > 0$$

which impose constraints not only on mass distribution and geometry of the body surface but, also, on the sign of angular velocity. The last condition, then coincides with the similar condition of stability of body rotation on an absolutely rough plane, and the first four differ from the respective conditions.

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